

Rainbow matchings and algebras of sets*

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Abstract

Grinblat (2002) asks the following question in the context of algebras of sets: What is the smallest number $\mathfrak{v} = \mathfrak{v}(n)$ such that, if A_1, \dots, A_n are n equivalence relations on a common finite ground set X , such that for each i there are at least \mathfrak{v} elements of X that belong to A_i -equivalence classes of size larger than 1, then X has a rainbow matching—a set of $2n$ distinct elements $a_1, b_1, \dots, a_n, b_n$, such that a_i is A_i -equivalent to b_i for each i ?

Grinblat has shown that $\mathfrak{v}(n) \leq 10n/3 + O(\sqrt{n})$. He asks whether $\mathfrak{v}(n) = 3n - 2$ for all $n \geq 4$. In this paper we improve the upper bound (for all large enough n) to $\mathfrak{v}(n) \leq 16n/5 + O(1)$.

1 Introduction

In this paper we attack a combinatorial problem previously raised and studied by Grinblat in [3, 4, 5]. Grinblat’s question arose in the study of algebras of sets. Given a nonempty ground set Y , let $\mathcal{P}(Y)$ denote the power set of Y . An *algebra (of sets)* on Y is a nonempty family $A \subseteq \mathcal{P}(Y)$ such that: (1) if $M \in A$ then also $Y \setminus M \in A$; and (2) if $M_1, M_2 \in A$ then also $M_1 \cup M_2 \in A$. Grinblat investigated necessary and sufficient conditions under which the union of at most countably many algebras on Y equals $\mathcal{P}(Y)$.

In this context, Grinblat asks for the smallest integer $\mathfrak{v} = \mathfrak{v}(n)$ for which the following holds: “Suppose A_1, \dots, A_n are n algebras on Y , such that, for each i , there exists a collection of at least \mathfrak{v} pairwise disjoint subsets of Y that are *not* in A_i . Then there exists a family $\{U_1, V_1, \dots, U_n, V_n\}$ of $2n$ pairwise-disjoint subsets of Y such that, for each i and each $Q \subseteq Y$, if Q contains one of U_i, V_i and is disjoint from the other, then $Q \notin A_i$.”

Grinblat shows that this problem is equivalent to the combinatorial problem presented below. (The equivalence is quite straightforward if the ground set Y is finite, but if Y is infinite then the argument is more delicate. We refer the reader to [5] for more details.)

1.1 The combinatorial problem

The combinatorial problem that interests us is the following: Let n be a positive integer. Let X be a finite ground set, and let A_1, \dots, A_n be n equivalence relations on X (or equivalently, partitions of X into subsets). If $a, b \in X$ are equivalent under A_i , then we say for short that a, b are i -equivalent, and we write $a \sim_i b$. The i -equivalence class of an element $a \in X$ is given

*An extended abstract of this paper appeared in *Eurocomb 2015 (Electronic Notes in Discrete Mathematics 49:251–257, 2015)*.

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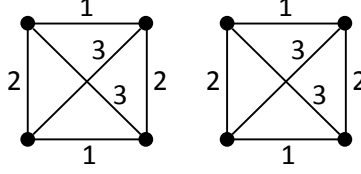


Figure 1: Here $|K_i| = 8$ for all $i = 1, 2, 3$, and yet there is no rainbow matching.

by $[a]_i = \{b \in X : a \sim_i b\}$. The *kernel* of A_i , denoted K_i , is defined as the set of elements of X that are i -equivalent to some element other than themselves:

$$K_i = \{a \in X : |[a]_i| > 1\}.$$

(It will become evident that one can assume without loss of generality that all equivalence classes in each A_i have size at most 3.)

We shall call a set of $2n$ *distinct* elements $a_1, b_1, \dots, a_n, b_n \in X$ a *rainbow matching* if $a_i \sim_i b_i$ for each i . (See e.g. Glebov et al. [2] for the term.)

The problem is to find the smallest integer $\mathfrak{v} = \mathfrak{v}(n)$ such that, if $|K_i| \geq \mathfrak{v}$ for all i , then A_1, \dots, A_n have a rainbow matching.

Grinblat observed that $\mathfrak{v}(n) \geq 3n - 2$: If we let all equivalence relations A_i be identical, consisting of $n - 1$ equivalence classes of size 3, then they have no rainbow matching even though $|K_i| = 3n - 3$.

Grinblat also showed that $\mathfrak{v}(3) = 9$. The lower bound $\mathfrak{v}(3) > 8$ is illustrated in Figure 1.

Grinblat proved in [5] that $\mathfrak{v}(n) \leq \left\lceil 10n/3 + \sqrt{2n/3} \right\rceil$ (he previously announced a slightly weaker bound in [4]). He asks whether $\mathfrak{v}(n) = 3n - 2$ for all $n \geq 4$.

In this paper we improve the upper bound to $\mathfrak{v}(n) \leq 16n/5 + O(1)$:

Theorem 1. *Let A_1, \dots, A_n be n equivalence relations with kernels K_1, \dots, K_n , respectively. Suppose $|K_i| \geq (3 + 1/5)n + c$ for each i , where c is a large enough constant. Then A_1, \dots, A_n have a rainbow matching.*

As we will see, it is enough to take $c = 5000$ in Theorem 1.

1.2 Overview of the proof

We prove Theorem 1 by a modification of Grinblat's argument. The proof follows by induction on the number of equivalence relations n , showing that given a rainbow matching (of $n - 1$ pairs) for the equivalence relations A_2, \dots, A_n , it is possible to obtain a rainbow matching for A_1, A_2, \dots, A_n . The proof follows in three main steps, where in each step, we observe that it is either possible to complete a full fledged rainbow matching, or to slightly extend the previous formation at hand. The final formation, which is the result of the third step, allows us to complete the rainbow matching, concluding the proof.

As mentioned above, we start with a rainbow matching over A_2, \dots, A_n of the form $\{a_i, b_i\}_{i=2}^n$, where $a_i \sim_i b_i$. In the first step, we transform the first $t - 1 = \lfloor n/5 \rfloor$ pairs into a *track* of $t - 1$ *components* of the form $\{a_i, b_i, c_i, d_i\}_{i=2}^t$, where $a_i \sim_{i-1} c_i$ and $b_i \sim_{i-1} d_i$ (the choice of indices is without loss of generality, by some renaming of indices). This $(t - 1)$ -long track is called C_{left} and the remaining $4n/5$ pairs are called C_{right} (see Figure 2). The

construction of C_{left} is inductive, and follows by showing that an m -long track, for $m \leq t - 2$, can be extended. This is true, since otherwise, there exists a pair $x \sim_{m+1} y$ that allows us to complete a full fledged rainbow matching. See Lemma 3 below for the complete argument.

In the second step, we consider the elements of K_1 and K_t . Specifically, we count the number of such elements that are connected to a component C_i in C_{left} or to a component C_j in C_{right} . We first observe that no such C_i or C_j can account for more than four elements of either K_1 and K_t . Otherwise, it is not hard to complete a full fledged matching. We call a component *heavy* if it accounts for at least 7 elements in $K_1 \cup K_t$. We further observe that the existence of 5 heavy components in C_{left} enables us to complete the desired matching. We, hence, assume that all but four heavy components appear in C_{right} . Let H be the set of heavy components in C_{right} . A simple counting argument yields that H is of size at least $n/5 + c - 4$. Assuming that we cannot complete a rainbow matching using a single component C_i with $i \in H$, we move on to the following third step.

The third and final step pinpoints a component $C_{j^*} = \{a_{j^*}, b_{j^*}\}$ in C_{right} , such that there exist $u_i, v_i \in K_i$ with $a_{j^*} \sim_i u_i$ and $b_{j^*} \sim_i v_i$ for many $i \in H$. Furthermore, we prove the existence of two indices $i_1, i_2 \in H$, and a “free” pair of elements $x \sim_{j^*} y$, such that the following substitutions are possible, completing a full fledged rainbow matching: $x \sim_{j^*} y$ will represent A_{j^*} (replacing C_{j^*}), the two pairs representing A_{i_1}, A_{i_2} will be replaced with, say, $a_{j^*} \sim_{i_1} u_{i_1}$ and $b_{j^*} \sim_{i_2} v_{i_2}$, and finally, A_1 and A_t will now be represented by, say, $a_{i_1} \sim_1 q$ and $a_{i_2} \sim_t p$, where q, p exist since $i_1, i_2 \in H$ (see Figure 7 for an illustration).

1.3 Subsequent work

In a follow-up paper, Clemens et al. [1] have solved the problem asymptotically, by showing that $\mathfrak{v}(n) \leq 3n + O(\sqrt{n})$.

2 Proof of Theorem 1

Suppose that $|K_i| \geq (3 + 1/5)n + c$ for each $i = 1, \dots, n$, for some constant c to be specified later. We can assume by induction on n that A_2, \dots, A_n have a rainbow matching $a_2 \sim_2 b_2, a_3 \sim_3 b_3, \dots, a_n \sim_n b_n$. Let $B = \{a_2, b_2, \dots, a_n, b_n\}$.

Observation 2. *If there exist two distinct elements $a_1 \sim_1 b_1$ with $a_1, b_1 \in X \setminus B$ then we are immediately done.*

Hence, let us assume that the above is not the case. Thus, every element in $K_1 \setminus B$ must be 1-equivalent to some element of B (possibly more than one). However, no two distinct elements of $K_1 \setminus B$ can be 1-equivalent to the *same* element of B (by the transitivity of \sim_1).

Therefore, by the pigeonhole principle, there must exist an index¹ in $\{2, \dots, n\}$, which without loss of generality we assume to be 2, for which there exist two distinct elements $c_2, d_2 \in X \setminus B$ satisfying $a_2 \sim_1 c_2, b_2 \sim_1 d_2$.

We can now similarly consider K_2 : Unless we are immediately done, there must exist an index in $\{3, \dots, n\}$, which without loss of generality we assume to be 3, for which there exist two distinct elements $c_3, d_3 \in X \setminus (B \cup \{c_2, d_2\})$ satisfying $a_3 \sim_2 c_3, b_3 \sim_2 d_3$.

We can continue in this way:

¹Actually, many.

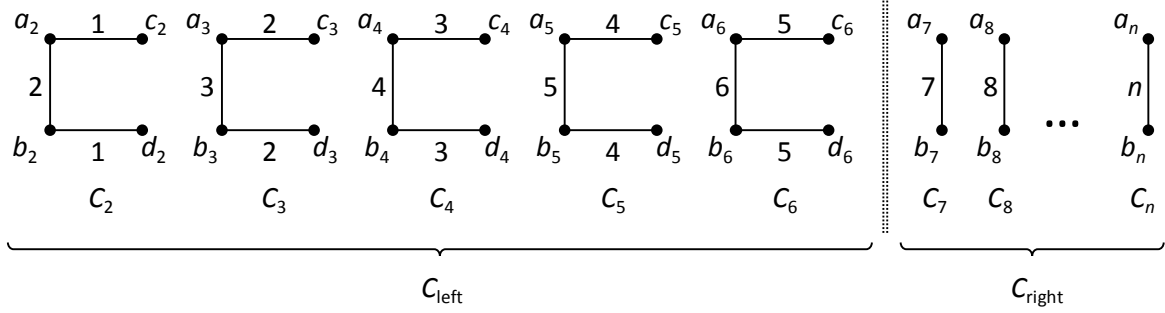


Figure 2: Left-side and right-side components.

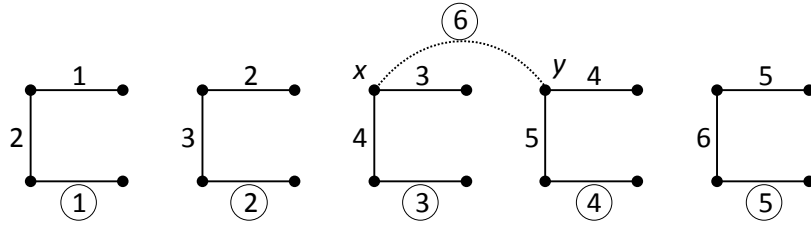


Figure 3: If $x \sim_i y$ (here $i = 6$), and x, y belong to different left-side components, or they both belong to the top (or the bottom) part of the same left-side component (case not shown), then we can easily complete a rainbow matching (indicated by the circled edge labels).

Lemma 3. Let $t = \lfloor n/5 \rfloor$. We can find $t - 1$ distinct indices in $\{2, \dots, n\}$, which without loss of generality we assume to be $2, \dots, t$, and we can find $2(t - 1)$ pairwise distinct elements $c_2, d_2, \dots, c_t, d_t \in X \setminus B$, such that $a_i \sim_{i-1} c_i$ and $b_i \sim_{i-1} d_i$ for all $2 \leq i \leq t$.

Proof. Suppose by induction that we have already found $c_2, d_2, \dots, c_i, d_i$.

Let $B' = B \cup \{c_2, d_2, \dots, c_i, d_i\}$. Partition B' into “components” as follows: $C_2 = \{a_2, b_2, c_2, d_2\}$, \dots , $C_i = \{a_i, b_i, c_i, d_i\}$; $C_{i+1} = \{a_{i+1}, b_{i+1}\}$, \dots , $C_n = \{a_n, b_n\}$. Let $C_{\text{left}} = C_2 \cup \dots \cup C_i$ and $C_{\text{right}} = C_{i+1} \cup \dots \cup C_n$. See Figure 2.

Observation 4. If there exist two distinct elements $x \sim_i y$, with $x, y \in K_i \setminus C_{\text{right}}$, then we are easily done unless one of x, y belongs to $\{a_j, c_j\}$ and the other one belongs to $\{b_j, d_j\}$ for the same index $2 \leq j \leq i$. See Figure 3.

Hence, let us charge each element of K_i to exactly one component, as follows:

Charging Scheme 1. Let $x \in K_i$. If $x \in B'$, then x is charged to the component it belongs to. Otherwise, by Observation 4, x must be i -equivalent to some $y \in C_{\text{right}}$; then we charge x to y ’s component. (If x can be charged to more than one right-side component, then we choose one of them arbitrarily.)

The total number of charges is equal to $|K_i|$, which is at least $(3 + 1/5)n + c$. By Observation 4 and the transitivity of \sim_i , no component can get more than four charges. Hence, if $i \leq n/5$, then there must be a component in C_{right} that received four charges. Without loss of generality it is C_{i+1} . Of the four elements charged to it, the two not belonging to it are the desired c_{i+1}, d_{i+1} .

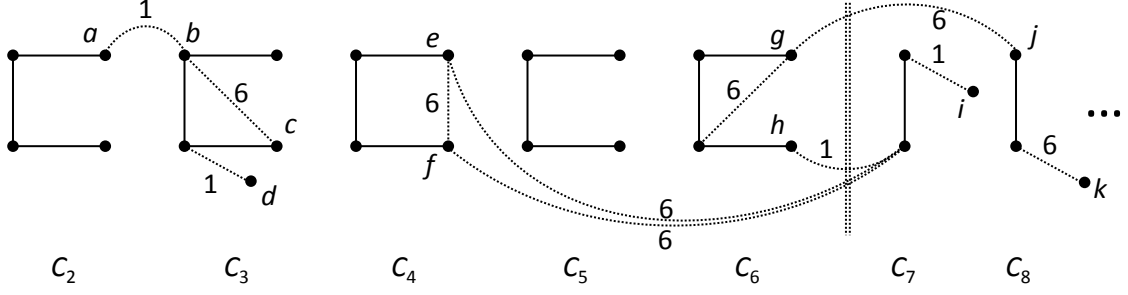


Figure 4: Charging Scheme 2. Here $t = 6$. Elements that belong to B (such as b and j) are always 1- or 6-charged to their own components. Elements a and d are 1-charged to C_3 . Elements h and i are 1-charged to C_7 . Element c is 6-charged to C_3 . Elements e and f are 6-charged to C_4 (despite the 6-edges to C_7). Element g is 6-charged either to C_6 or to C_8 . Element k is 6-charged to C_8 .

This concludes the proof of Lemma 3. \square

Define the set B' , the components C_2, \dots, C_n , and the sets C_{left} and C_{right} as above, with t in place of i . Hence, $C_{\text{left}} = C_2 \cup \dots \cup C_t$ and $C_{\text{right}} = C_{t+1} \cup \dots \cup C_n$.

We now use the following charging scheme for A_1 and A_t :

Charging Scheme 2. Consider an element $x \in K_1$. If $x \in B$, then we 1-charge x to the component it belongs to. Otherwise, by Observation 2, x must be 1-equivalent to some element $y \in B$; then we 1-charge x to the component that contains y .

Consider the elements of K_t . We t -charge every element $z \in (K_t \cap B)$ to the component it belongs to. If $c_i \sim_t d_i$ for some i , then we t -charge both elements to the component C_i that contains them. For every $z \in K_t$ not covered by the above cases, by Observation 4 there must be a component C_i that contains an element $y \sim_t z$ (furthermore, either C_i is a right-side component, or else $z \in C_i$); we charge z to C_i .

Figure 4 illustrates Charging Scheme 2.

Lemma 5. *In Charging Scheme 2, no component C_i can receive more than four 1-charges, or more than four t -charges.*

Proof. Since no two different elements outside B can be 1-equivalent to the same element of B , every component C_i can receive at most two 1-charges from its own elements a_i, b_i , plus at most two more 1-charges from other elements.

The argument regarding t -charges is only slightly more complicated: There *might* be an element z in a right-side component C_i that is t -equivalent to two different elements outside B . However, then they must be c_j, d_j for some left-side component C_j , by Observation 4. Hence, they are t -charged to C_j and not to C_i . \square

For each $2 \leq i \leq n$, let σ_i (resp. τ_i) be the number of 1-charges (resp. t -charges) that component C_i received; let S_i (resp. T_i) be set of elements *not* in $\{a_i, b_i\}$ that were 1-charged (resp. t -charged) to C_i ; and let $U_i = S_i \cup T_i$.

The sets S_i are by definition pairwise disjoint, as are the sets T_i . However, for a fixed i , S_i is not necessarily disjoint from T_i ; and U_i is not necessarily disjoint from $B' \setminus B$. Still, no three sets U_i have a common intersection.

Lemma 5 states that $\sigma_i \leq 4$ and $\tau_i \leq 4$ for each i . Furthermore, by the argument in the proof of Lemma 5, we have $|S_i| \leq 2$ and $|T_i| \leq 2$. Moreover, we have $\sum \sigma_i = |K_1|$ and $\sum \tau_i = |K_t|$, each of which is at least $(3 + 1/5)n + c$.

Lemma 6. *Suppose that there exist five different left-side components that receive at least 7 charges each; namely, suppose there exist C_{i_1}, \dots, C_{i_5} , with $2 \leq i_1 < \dots < i_5 \leq t$, such that $\sigma_{i_k} + \tau_{i_k} \geq 7$ for each $1 \leq k \leq 5$. Then we can complete a rainbow matching.*

Proof. Consider the component C_{i_2} . For simplicity rename its four elements a', b', c', d' in the obvious way. Since this component received at least three t -charges, there must be a pair of elements among a', b', c', d' that are t -equivalent *excluding* the pair $\{a', b'\}$. This pair cannot be $\{a', c'\}$ nor $\{b', d'\}$, by Observation 4. Hence, the pair must be $\{c', d'\}$ (case 1), or $\{a', d'\}$ or $\{b', c'\}$ (case 2).

In case 1, we consider components $C_{i_3}, C_{i_4}, C_{i_5}$. In each one of them there must be an a_j or b_j that is 1-equivalent to some $y \notin B$. At most two of these y 's can be c' or d' , so the third one leads to a win, as follows (see Figure 5, top): Suppose for concreteness that $a_{i_5} \sim_1 y$ for $y \notin \{c', d'\}$. Then we take the pairs $c' \sim_t d'$, $a_{i_5} \sim_1 y$; the pairs $a_i \sim_i b_i$ for all $2 \leq i < i_5$; and the pairs $a_i \sim_{i-1} c_i$ for all $i_5 < i \leq t$, except that if $y = c_i$ for some $i > i_5$ then we take $b_i \sim_{i-1} d_i$ instead.

Now consider case 2. Suppose for concreteness that $a' \sim_t d'$ (the case $b' \sim_t c'$ is symmetric). Let us look again at component C_{i_2} . First suppose that it received four 1-charges. Then each of a', b' must be 1-equivalent to an element not in B . Consider the element $y \sim_1 b'$ that is not in B . If $y \neq d'$ then we go to case 2a below; if $y = d'$ then we go to case 2b below.

Now suppose C_{i_2} received four t -charges. Then we must have *both* $a' \sim_t d'$ and $b' \sim_t c'$. Furthermore, C_{i_2} received at least three 1-charges, so at least one of a', b' , say b' , must be 1-equivalent to some $y \notin B$. As before, if $y \neq d'$ we go to case 2a; otherwise we go to case 2b.

Case 2a is an easy win by taking the pairs $b' \sim_1 y$ and $a' \sim_t d'$, and completing the rainbow matching as in Figure 5 (middle).

In case 2b, we take the pairs $a' \sim_{i_2-1} c'$ and $b' \sim_1 d'$; from C_{i_1} we take a pair $x \sim_t y$; and we complete the rainbow matching as in Figure 5 (bottom). \square

Recall that no component can receive more than 8 charges, and that the total number of charges is $2(16n/5 + c)$. Therefore, the number of components that receive at least 7 charges must be at least $n/5 + c$. Lemma 6 implies that at most four of these components can be on the left side.

Hence, there must be at least $n/5 + c - 4 \geq n/5$ right-side components that receive at least 7 charges each. Call such components “heavy”, and let H be the set of their indices; namely, let

$$H = \{i \in \{t+1, \dots, n\} : \sigma_i + \tau_i \geq 7\}.$$

For each $i \in H$ we have $|S_i| \geq 1$, $|T_i| \geq 1$, $\max\{|S_i|, |T_i|\} = 2$, and $2 \leq |U_i| \leq 4$. Furthermore, for every two distinct heavy indices $i, j \in H$ we have $|U_i \cup U_j| \geq 3$.

Let $i \in H$. It is not necessarily possible to find four distinct elements $v_1, v_2 \in C_i$, $w_1, w_2 \in U_i$, such that $v_1 \sim_1 w_1$ and $v_2 \sim_t w_2$ (even if $\sigma_i = \tau_i = 4$) since we could have $a_i \sim_1 x$, $b_i \sim_1 y$, $a_i \sim_t y$, $b_i \sim_t x$. Nevertheless, we can prove the following lemma:

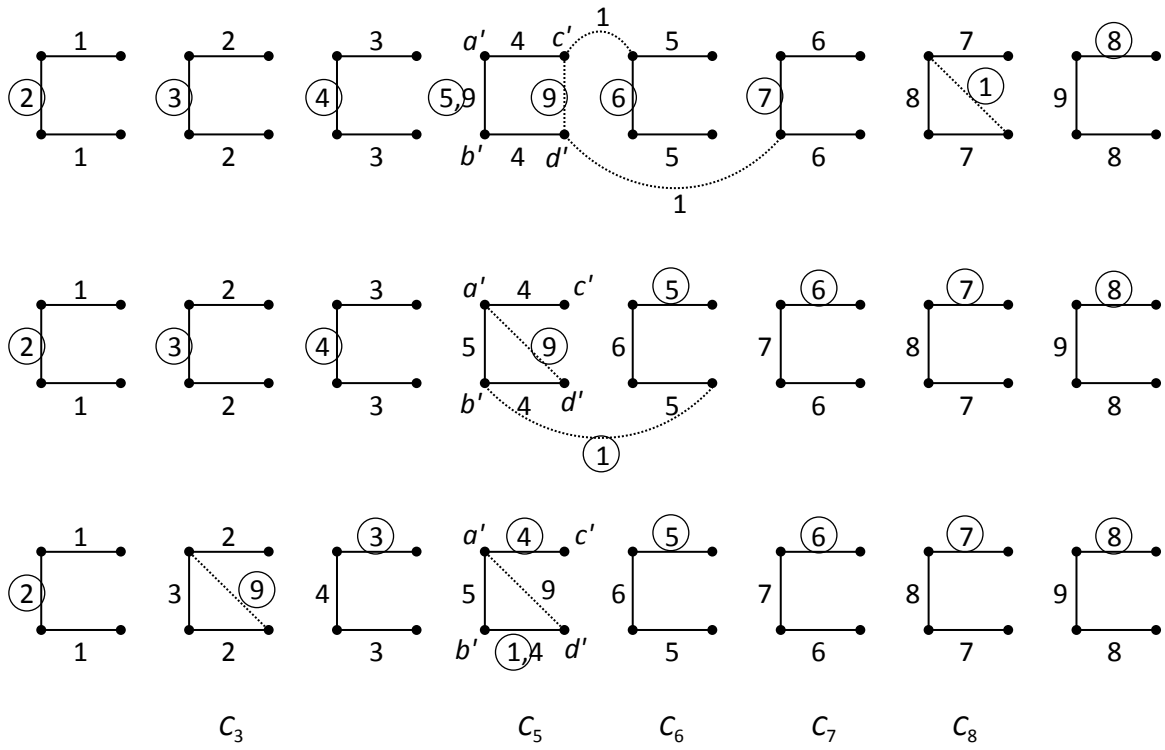


Figure 5: Three cases considered in the proof of Lemma 6. Here $t = 9$, and the components that receive 7 charges are C_3 , C_5 , C_6 , C_7 , C_8 .

Lemma 7. *Let C_i, C_j , $i, j \in H$, be two distinct heavy components. Then we can find four distinct elements $v_1, v_2 \in C_i \cup C_j$, $w_1, w_2 \in U_i \cup U_j$, such that $v_1 \sim_1 w_1$ and $v_2 \sim_t w_2$.*

Furthermore, for any two fixed elements $q, r \in U_i \cup U_j$, it is always possible to do so guaranteeing that exactly one of q, r belongs to $\{w_1, w_2\}$.

The “furthermore” clause of Lemma 7 will be used once, in the proof of Lemma 12 below. Unfortunately, it requires a tedious case analysis.

Proof of Lemma 7. Suppose first that $q \in S_i \cap T_i$ (hence, $q \notin S_j$ and $q \notin T_j$, so $q \notin U_j$). We have $|U_j| \geq 2$, so there exists an element $s \in U_j \setminus \{r\}$. If $s \in S_j$ then we can take $w_1 = s$, $w_2 = q$ and finish; otherwise, $s \in T_j$, so we are done by taking $w_1 = q$, $w_2 = s$.

The case $q \in S_j \cap T_j$ is symmetric, as well as the cases $r \in S_i \cap T_i$ and $r \in S_j \cap T_j$. So suppose none of these cases apply.

Suppose for concreteness that $q \in S_i$ (the three other possibilities are symmetric). Consider T_j . If it contains an element $s \notin \{q, r\}$, then we are done by taking $w_1 = q$, $w_2 = s$. Hence, assume $T_j \subseteq \{q, r\}$.

Suppose $q \in T_j$. Suppose for concreteness that $q \sim_1 a_i$ and $q \sim_t a_j$. Then b_i must be 1- or t -equivalent to an element $z_1 \in U_i$, $z_1 \neq q$; and b_j must be 1- or t -equivalent to an element $z_2 \in U_j$, $z_2 \neq q$. If one of z_1, z_2 is different from r , then we are done by taking it and q for $\{w_1, w_2\}$. Otherwise, we have $r = z_1 = z_2$. Take a third element $s \in (U_i \cup U_j) \setminus \{q, r\}$. Hence, s is 1- or t -equivalent to one of a_i, a_j, b_i, b_j . In the first two cases we take $\{w_1, w_2\} = \{r, s\}$, whereas in the last two cases we take $\{w_1, w_2\} = \{q, s\}$.

Finally, suppose $T_j = \{r\}$. Say r is t -equivalent to a_j . Then b_j must be 1- or t -equivalent to some element $s \notin \{q, r\}$. Then we take $\{w_1, w_2\}$ to be s and one of q, r . \square

For $i \in H$, let us call a left-side component i -tainted if it intersects T_i . Since $|T_i| \leq 2$, for each i there are at most two i -tainted components.

Lemma 8. *Let C_i , $i \in H$ be a heavy component.*

- (a) *If there exist two distinct elements $x \sim_i y$, both outside $B \cup S_i$, then we are done.*
- (b) *Let C_j be a left-side component that is not i -tainted. Then, if one of a_j, b_j is i -equivalent to an element z outside $B' \cup T_i$, then we are done.*

Proof. In case (a), take an element $u \in S_i$. Note that $u \notin \{x, y\}$ by assumption. Proceed as in Figure 6(a).

In case (b), take an element $v \in T_i$. Note that $v \notin \{z\} \cup C_j$ by assumption. Proceed as in Figure 6(b). \square

Recall that $|H| \geq n/5$. Fix an index $i \in H$, and consider the set $L_i = K_i \setminus (B' \cup U_i)$. By Lemma 8(a), each element of L_i must be i -equivalent to a *different* element of $B \cup S_i$ (by transitivity of \sim_i). Hence, there are at most two elements of L_i that are i -equivalent to elements of S_i , and at most four more that are i -equivalent to a_j or b_j in an i -tainted component C_j . All the remaining elements of L_i must be i -equivalent to elements of C_{right} , by Lemma 8(b).

Hence, let us i -charge the elements of $K_i \setminus U_i$ to components according to the following charging scheme (which is similar to Charging Scheme 1):

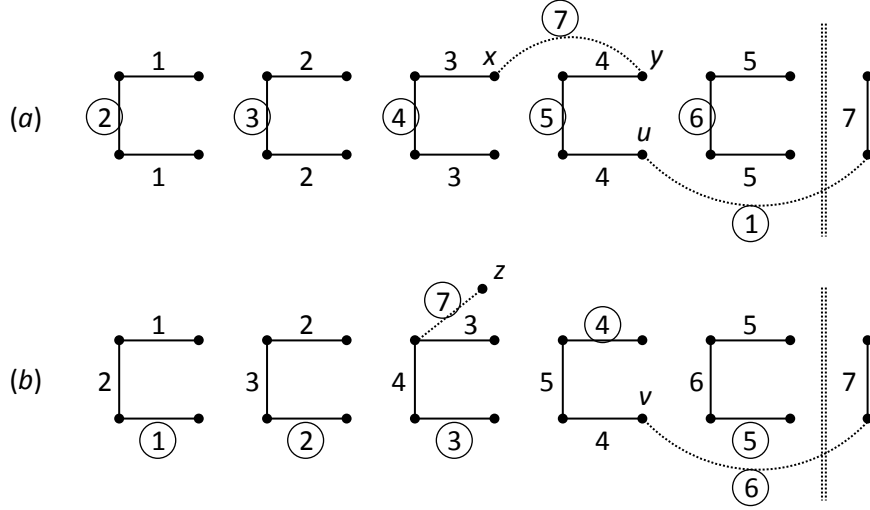


Figure 6: Proof of Lemma 8. Here, $t = 6$ and $i = 7$, and in case (b), $j = 4$.

Charging Scheme 3. Let $i \in H$. Consider an element $x \in K_i \setminus U_i$. If $x \in B'$, then x is i -charged to the component it belongs to. Otherwise, if x is i -equivalent to an element of S_i or to a_j or b_j where component C_j is i -tainted, then x is not i -charged. Otherwise, x must be i -equivalent to an element $y \in C_{\text{right}}$; then we charge x to the component that contains y .

Lemma 9. *In Charging Scheme 3, no component receives more than four i -charges.*

Proof. By the above considerations, left-side components only receive charges from their own elements, and right-side components can receive at most two outside charges. \square

We have $|K_i \setminus U_i| \geq |K_i| - 4$, and there are at most six elements of this set that are not charged. Hence, there are at least $n/5 + c - 10$ components that received at least four charges. Out of them, at least $c - 10$ are right-side components.

Let us apply this charging for all $i \in H$. By the pigeonhole principle, there must be a “lucky” right-side component C_{j^*} that receives four i -charges for all $i \in H'$, for some subset $H' \subseteq H$ of size $|H'| = ((c - 10)n/5)/(4n/5) = (c - 10)/4$. For each index $k \in H'$, let W_k be the set of two elements not in C_{j^*} that were k -charged to C_{j^*} . Note that each W_k is disjoint from B' , since elements of B' were charged to their own components. Furthermore, W_k is disjoint from U_k for each k (since the elements of U_k were not used).

Let k_1, k_2 be a pair of distinct indices in H' . We would like to choose four distinct elements w, x, y, z , with $C_{j^*} = \{w, x\}$ and $y, z \in W_{k_1} \cup W_{k_2}$, such that $w \sim_{k_1} y$ and $x \sim_{k_2} z$. If such a choice is not possible, then call the pair k_1, k_2 “conflicting”.

Lemma 10. *There exists a subset $H'' \subset H'$, of size $|H''| = c/16$, such that no two indices in H'' are conflicting.*

Proof. The only way for k_1, k_2 to be conflicting is to have $C_{j^*} = \{w, x\}$, $W_{k_1} = W_{k_2} = \{y, z\}$, $w \sim_{k_1} y$, $x \sim_{k_1} z$, $w \sim_{k_2} z$, $x \sim_{k_2} y$. Therefore, if we define an undirected graph having H' as vertex set, and having an edge $k_1 k_2$ whenever k_1, k_2 are conflicting, this graph cannot have an odd cycle, and therefore it is bipartite, and hence it has an independent set of size at least

$|H'|/2 = (c - 10)/8$. Out of this independent set, we select an arbitrary subset H'' of size $c/16$. \square

Let $H'' \subset H'$ be as in Lemma 10, and let $W = \bigcup_{k \in H''} W_k$. We do not have a good handle on the size of W ; it could be anything in the range $2 \leq |W| \leq 2|H''|$.

Let k_1, k_2 be a pair of distinct indices in H'' . We already know that k_1, k_2 are not conflicting, and that $W_{k_1} \cap U_{k_1} = W_{k_2} \cap U_{k_2} = \emptyset$. We would also like to have $W_{k_1} \cap U_{k_2} = W_{k_2} \cap U_{k_1} = \emptyset$. If that is the case, call the pair k_1, k_2 “compatible”.

Lemma 11. *Let $c = 5000$. Then there exists a compatible pair of distinct indices $k_1, k_2 \in H''$.*

Proof. Call an element of W “popular” if it appears in at least \sqrt{c} sets W_k , $k \in H''$. Let $W_p \subseteq W$ be the set of popular elements; we have $|W_p| \leq 2|H''|/\sqrt{c} \leq \sqrt{c}/8$.

Since each element can appear in at most two sets U_k (at most one S_k and at most one T_k), there are at least

$$|H''| - 2|W_p| \geq c/16 - \sqrt{c}/4 \geq 1$$

indices $k_1 \in H''$ for which U_{k_1} contains no popular element.

Pick one such index k_1 . The elements of U_{k_1} appear in at most $4\sqrt{c}$ sets W_{k_2} , and the elements of W_{k_1} appear in at most 4 other sets U_{k_2} . That leaves us with at least

$$c/16 - 4\sqrt{c} - 4 \geq 1$$

choices for $k_2 \in H''$ that make the pair k_1, k_2 compatible. \square

We are almost done:

Lemma 12. *If there exist two distinct elements $x \sim_{j^*} y$, both outside*

$$Y = C_{\text{right}} \cup W \cup \left(\bigcup_{k \in H''} U_k \right),$$

then we are done.

Proof. Complications arise only when $\{x, y\} \subset C_{\text{left}}$.

Suppose first that x and y belong to the same left-side component. By Lemma 11, let $k_1, k_2 \in H''$ be a compatible pair of distinct indices. Invoke Lemma 7 with $i = k_1$, $j = k_2$. We obtain four distinct elements $v_1 \sim_1 w_1$, $v_2 \sim_t w_2$, with $v_1, v_2 \in C_{k_1} \cup C_{k_2}$ with $w_1, w_2 \in U_{k_1} \cup U_{k_2}$. If we are unlucky and $\{w_1, w_2\} = \{c_j, d_j\}$ for some index j , then we invoke the “furthermore” clause of Lemma 7 with $\{q, r\} = \{w_1, w_2\}$, and we proceed as in Figure 7(a).

Now suppose x and y belong to different left-side components C_i and C_j . Suppose without loss of generality that $x \in \{a_i, c_i\}$ and $y \in \{a_j, c_j\}$. Then we take a $k \in H''$ such that $d_i, d_j \notin T_k$. Such a k must exist, since $|H''| \geq 3$. Then we proceed as in Figure 7(b). \square

But a pair x, y as in Lemma 12 must exist, since otherwise, every element of K_{j^*} would either belong to Y or be j^* -equivalent to a different element of Y . Observe that $|Y| \leq 2(4/5)n + c/8 + c/4$. That accounts for only $2|Y| \leq (3 + 1/5)n + 3c/4$ elements of K_{j^*} , which is not enough. This concludes the proof of Theorem 1. \square

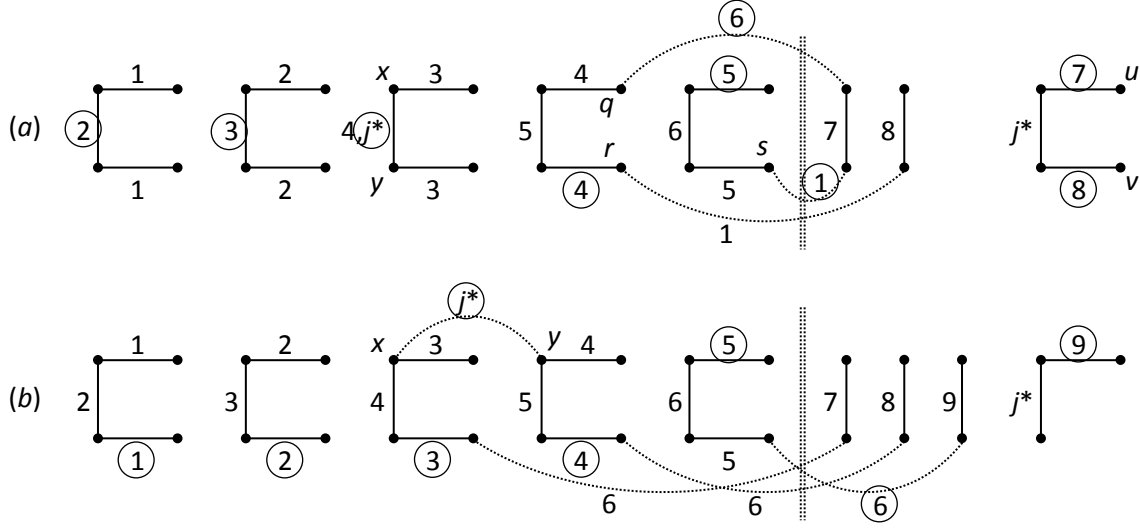


Figure 7: Proof of Lemma 12. Here $t = 6$. In case (a) we have $\{k_1, k_2\} = \{7, 8\}$. Since the indices k_1, k_2 are nonconflicting and compatible, the elements q, s, u, v are all distinct. In case (b) we have $H'' \supset \{7, 8, 9\}$ and $k = 9$.

Remark 1. The main difference between Charging Scheme 2 on the one hand, and Charging Schemes 1 and 3 on the other hand, is the way they handle the elements of $B' \setminus B$. In Charging Schemes 1 and 3 these elements are automatically charged to the component they belong to, whereas in Charging Scheme 2 they are not. If we modified Charging Scheme 3 to be like Charging Scheme 2 in this respect, then no left-side component would receive more than two i -charges, improving Lemma 9. However, we then run into trouble since the sets W_k might intersect $B' \setminus B$.

Acknowledgements Thanks to L. Š. (Yehuda) Grinblat for suggesting us to look at this problem and for helpful discussions. Special thanks to the referees for reading the paper carefully and providing detailed suggestions. Thanks also to Anat Paskin-Cherniavsky for helpful discussions.

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